



Quantum Grammars. Part IV.1: KMS States on Quantum Grammars

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***Quantum Grammars. Part IV.1: KMS states on
quantum grammars.***

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N° 3702

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Quantum Grammars. Part IV.1: KMS states on quantum grammars.

Vadim Malyshev

Thème 1 — Réseaux et systèmes
Projet Meval

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Abstract: This is the fourth part of the series of papers devoted to some relationships between the computer science and the quantum gravity. We consider quantum (unitary) continuous time evolution of spins on a lattice together with quantum evolution of the lattice itself. In physics such evolution was discussed in connection with the quantum gravity. It is also related to what is called quantum circuits, one of the incarnations of the quantum computers. We consider simpler models for which one can obtain exact mathematical results. We prove existence of the dynamics in both Schroedinger and Heisenberg pictures, construct KMS states on appropriate C^* -algebras.

We show (for high temperatures) that for each system where the lattice undergoes quantum evolution, there is a natural scaling leading to a quantum spin system on a fixed lattice Z , defined by a renormalized Hamiltonian.

Key-words: quantum computer, quantum gravity, quantum space, particle spectrum, temperature states, grammar, graph grammar

(Résumé : *tsvp*)

Grammaires Quantiques. Partie IV.1.

Résumé : On considère l'espace quantique rempli avec des particules quantiques comme un ordinateur quantique. Par changement d'échelle on retrouve l'espace classique avec des spins quantiques.

Mots-clé : ordinateur quantique, gravité quantique, grammaire

1 Introduction

Practical quantum computation has not yet started but many standard notions of the computer science have already been generalized, giving rise to the quantum computer science, see recent reviews [33, 35, 34]. Here we give a definition of a quantum grammar similar to the definition of a random grammar, given in [1].

A very particular case of quantum grammars are quantum spin systems, popular standard models in statistical physics and quantum field theory. Quantum grammar can be considered as a quantum spin system on a quantum lattice, that is the lattice itself is a quantum object subject to a unitary evolution. It is quite in a spirit of some approaches to the quantum gravity, where space is quantized, but the time remains classical and one-dimensional.

Here we consider questions pertinent to physical systems rather than to the computer science. We show how standard quantum spin systems (spin represents the matter) on the lattice Z (lattice represents the space) can emerge from KMS states on the C^* -algebras corresponding to quantum grammars. The term grammar refers normally to one-dimensional systems. Higher dimensional objects are called graph grammars in computer science. Higher dimension means only that it is not one-dimensional. The terms spin graph, spin complex or spin network are used instead of "higher dimensional grammars".

One of our goals is to show that already in one dimension these models have sufficiently interesting structure. The evolution of the space is simple however. There is no topology, only metrics is important: the space can expand and compress at any point, expanding and compressing being a quantum process. However there are phenomena which have no analogs in the statistical physics and quantum field theory living on a classical space.

The correspondence between grammars and quantum grammars are as between classical and quantum computation. We consider continuous time evolution which allows the grammars be far from context free. Thus there are no "no-go" theorems as for the discrete time, see [36]. We prove selfadjointness of the Hamiltonian which gives the unitary evolution on a Hilbert space and an automorphism group of some hyperfinite C^* -algebra. We show that there is a transition in the parameters (the temperature and the cosmological constant) when the KMS state exists or not. In the latter case we define renormalised KMS states, the scaling limit of such renormalized states is a standard quantum spin system.

2 Symmetric Grammars

2.1 Hilbert space and Hamiltonian

Let $S = \{1, \dots, r\}$ be a finite set (the alphabet), $L = L(S)$ - the set of all finite words (including the empty one) $\alpha = x_1 \dots x_n$, $x_i \in S$, in this alphabet. Length n of the word α is denoted by $|\alpha|$. Concatenation of two words $\alpha = x_1 \dots x_n$ and $\beta = y_1 \dots y_m$ is defined by

$$\alpha\beta = x_1 \dots x_n y_1 \dots y_m$$

The word β is a subword of α if there exist words δ and γ such that $\alpha = \delta\beta\gamma$. Grammar over S is defined by a finite set Sub of substitutions (productions), that is the pairs $\delta_i \rightarrow \gamma_i, i = 1, \dots, k = |Sub|, \delta_i, \gamma_i \in L$. Further on we assume that all δ_i, γ_i are not empty.

Let $\mathcal{H} = l_2(L)$ be the Hilbert space with the orthonormal basis $e_\alpha, \alpha \in L : (e_\alpha, e_\beta) = \delta_{\alpha\beta}$ where the function $e_\alpha(\beta) = \delta_{\alpha\beta}$. Each vector ϕ of \mathcal{H} a function on the set of words and can be written as

$$\phi = \sum \phi(\alpha) e_\alpha \in \mathcal{H}, \|\phi\|^2 = \sum |\phi(\alpha)|^2$$

States of the system are wave functions, that is vectors ϕ with the unit norm $\|\phi\|^2 = 1$. We shall define dynamics in the form

$$\phi(t) = \exp(itH)\phi(0)$$

The Hamiltonian H will be written in terms of operators, which resemble creation-annihilation operators in quantum field theory. For each $i = 1, \dots, k$ and each integer $j \geq 1$ we define quantum substitutions, that is linear bounded operators $a_i(j)$. If $\alpha = \tau\delta_i\rho$ for some words $\tau, \rho, |\tau| = j - 1$, we put

$$a_i(j)e_\alpha = e_\beta$$

where $\beta = \tau\gamma_i\rho$. Otherwise we put $a_i(j)e_\alpha = 0$. Adjoint operators $a_i^*(j)$ are defined by

$$a_i^*(j)e_\beta = e_\alpha$$

for $\beta = \tau\gamma_i\rho$ and 0 otherwise. Define the formal Hamiltonian by

$$H = \sum_{i=1}^{|Sub|} \sum_{j=1}^{\infty} (\lambda_i a_i(j) + \bar{\lambda}_i a_i^*(j))$$

for some complex λ_i .

We could equally assume that together with the substitution $\delta_i \rightarrow \gamma_i$ also its "inverse" substitution $\gamma_i \rightarrow \delta_i$ belongs to Sub . The Hamiltonian then can be written simply as

$$H = \sum_{i=1}^{|Sub|} \sum_{j=1}^{\infty} \lambda_i a_i(j)$$

We always assume that $H = H^*$, that is $\lambda_i = \bar{\lambda}_j$ in case $\delta_j = \gamma_i, \gamma_j = \delta_i$. We shall use only this representation further on.

H is well-defined and symmetric on the set $D(L)$ of finite linear combinations of e_α . These vectors are C^∞ -vectors for H , that is $He_\alpha \in D(L)$.

Theorem 1 H is essentially selfadjoint on $D(L)$.

Proof. We shall prove that each vector $\phi \in D(L)$ is an analytic vector of H , that is

$$\sum_{k=0}^{\infty} \frac{\|H^k \phi\|}{k!} t^k < \infty$$

for some $t > 0$. It is sufficient to take $\phi = e_\alpha$ for some α . Then the number of pairs (i, j) such that $a_i(j)e_\alpha \neq 0$ is not greater than $nk, n = |\alpha|, k = |Sub|$.

Write the decomposition of H as

$$H = \sum_a V_a$$

where V_a equals one of $\lambda_i a_i(j)$. Then

$$H^n e_\alpha = \sum_{a_n, \dots, a_1} V_{a_n} \dots V_{a_1} e_\alpha = \sum C_\beta e_\beta \quad (1)$$

The maximal length of the words β in the expansion of $V_{a_n} \dots V_{a_1} e_\alpha$ does not exceed $|\alpha| + C_1 n, C_1 = \max(|\gamma_i| - |\delta_i|)$. Then, for given $e_\alpha, a_1, \dots, a_n$, the number of operators $V_{a_{n+1}}$ giving a nonzero contribution to $V_{a_{n+1}} V_{a_n} \dots V_{a_1} e_\alpha$. It does not exceed $k(|\alpha| + C_1 n), k = |Sub|$. Thus the number of nonzero terms $V_{a_n} \dots V_{a_1} e_\alpha$ does not exceed

$$k^n \prod_{j=1}^n (|\alpha| + C_1 j) = (kC_1)^n \frac{(\frac{|\alpha|}{C_1} + n)!}{n! (\frac{|\alpha|}{C_1})!} \leq (kC_1)^n n^{\frac{|\alpha|}{C_1}}$$

and the norm of each term is bounded by $(\max \lambda_i)^n$. This gives convergence of the series for $|t| < t_0$ where t_0 does not depend on α .

2.2 C^* -algebra

For each N let $\mathcal{H}_N \subset \mathcal{H}$ be the finite dimensional subspace generated by all e_α with $|\alpha| \leq N$, let P_N be the orthogonal projection onto \mathcal{H}_N . Let \mathbf{A}_N be the C^* -algebra of all operators in \mathcal{H}_N . It is the $(\frac{r^{N+1}-1}{r-1} \times \frac{r^{N+1}-1}{r-1})$ -matrix algebra if $r > 1$. We can consider "cut-off" operators

$$a_{i,N}(j) = P_N a_i(j) P_N$$

as belonging to \mathbf{A}_N .

We have natural embeddings $\mathcal{H}_N \subset \mathcal{H}_{N+1}$ and we define the embeddings $\phi_N : \mathbf{A}_N \rightarrow \mathbf{A}_{N+1}$ by: for $B \in \mathbf{A}_N$ we put $\phi_N(B)e_\alpha = Be_\alpha$ if $|\alpha| \leq N$ and $\phi_N(B)e_\alpha = 0$ if $|\alpha| = N+1$. The inductive limit $\cup_N \mathbf{A}_N = \mathbf{A}^0$ of the C^* -algebras \mathbf{A}_N is called the local algebra, its norm closure \mathbf{A} is called the quasilocal algebra. It does not fall however under the general definition of quasilocal algebras [6], due to the absence of "space structure". There is no identity element in this algebra (it can be appended if necessary, the identity operator in

\mathcal{H}), but there is an approximate identity, a sequence $1(\mathbf{A}_N)$ of unit matrices in \mathbf{A}_N . \mathbf{A} is a hyperfinite C^* -algebra.

Note that the formal Hamiltonian H defines the differentiation of the local algebra. Denote

$$H_N = \sum_{i=1}^{|Sub|} \sum_{j=1}^N \lambda_i a_{i,N}(j)$$

Take some local A and N such that $A \in \mathbf{A}_N$. Define an automorphism group of \mathbf{A}_N as follows

$$\alpha_t^{(N)}(A) = \exp(iH_N t) A \exp(-iH_N t)$$

Theorem 2 *There exists $t_0 > 0$ such that for any local A and for each $t, |t| < t_0$, there exists the norm limit*

$$\lim_{N \rightarrow \infty} \alpha_t^{(N)}(A)$$

This defines a unique automorphism group of the quasilocal algebra.

Proof. Consider the Dyson-Schwinger series

$$A_t^{(N)} = A + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [H_N, \dots, [H_N, [H_N, A]] \dots]$$

One can take $A = A_{\alpha\rho}$ where $A_{\alpha\rho} e_\gamma = \delta_{\alpha\gamma} e_\rho$. Note that the commutator is the sum of commutators

$$[a_{i_n}(j_n), \dots, [a_{i_2}(j_2), [a_{i_1}(j_1), A]], \dots]$$

multiplied by $\lambda_{i_1} \dots \lambda_{i_n}$. Nonzero commutators should have the property that $j_k \leq l(A) + C_1(k-1)$, $l(A) = \max(|\alpha|, |\rho|)$. The convergence proof is quite similar to the previous convergence proof. If $N \rightarrow \infty$ then $A_t^{(N)}$ converge to

$$A_t = A + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [H, \dots, [H, [H, A]] \dots]$$

Each term of the latter series is well defined and the series converges for t sufficiently small. The existence of the automorphism group can be proved as in the Robinson theorem for quantum spin systems, see [6].

Remark 1 *Note that in the Robinson theorem for one-dimensional quantum spin systems with finite interaction radius one can prove that the series converges for all t , because the length of the cluster increases only at the boundary (that is at two end points). For quantum grammars this is not the case.*

2.3 KMS-states

To define temperature states on \mathbf{A} one could put for any local A and large N

$$\langle A \rangle_\beta = \lim_{N \rightarrow \infty} \langle A \rangle_{\beta, N} = \lim_{N \rightarrow \infty} Z_N^{-1} \text{Tr}_N [A \exp(-\beta H_N)], \quad Z_N = \text{Tr}_N \exp(-\beta H_N)$$

where Tr_N means the trace in \mathbf{A}_N . However, this does not always define a state. For example this gives zero for $\beta = 0$, where $Z_N = \frac{r^{N+1}-1}{r-1} \rightarrow \infty, r > 1, Z_N = N+1 \rightarrow \infty, r = 1$, but $\text{Tr}_N A$ is bounded. We shall prove it now in the general case but only for small β .

Lemma 1 *There exists $\beta_0 > 0$ such that for each local A the limit*

$$\lim_{N \rightarrow \infty} \text{Tr} [A \exp(-\beta H_N)]$$

exists and is analytic in β for $\beta < \beta_0$.

Proof. Take again $A = A_{\alpha\rho}$. Then

$$\text{Tr}_N [\exp(-\beta H_N) A] = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} \text{Tr}_N (H^k A) = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} \sum_{I_k, J_k} \text{Tr}_N (a_{i_k}(j_k) \dots a_{i_1}(j_1) A)$$

where the sum is over all arrays $I_k = (i_1, \dots, i_k), J_k = (j_1, \dots, j_k)$. The convergence proof is the same as for the previous statements.

Lemma 2 *If β is small then $\log Z_N \sim cN$ with $c > 1$. It follows that the above limit $\langle A \rangle_\beta$ is zero for all A .*

Proof. One can write

$$Z_N = \sum_{\alpha: |\alpha| \leq N} z(\alpha), \quad z(\alpha) = (e_\alpha, \exp(-\beta H_N) e_\alpha)$$

and

$$Z_N = \sum_{k=0}^{\infty} \sum_{I_k, J_k} \sum_{\alpha: |\alpha| \leq N} \frac{(-\beta)^k}{k!} z(\alpha, (I_k, J_k)), \quad z(\alpha, (I_k, J_k)) = (e_\alpha, a_{i_k}(j_k) \dots a_{i_1}(j_1) e_\alpha)$$

Consider some term of this expansion corresponding to some word α of length n and to some (I_k, J_k) . It is convenient to denote $\delta(p) \rightarrow \gamma(p)$ the substitution on place p , corresponding to the operator $b(p) = a_{i_p}(j_p)$. The symbol x_i of the word $\alpha = x_1 \dots x_n$ is called untouched for given (I_k, J_k) if no $\delta(p)$ contains it. Similarly, a symbol of the word $b(s) \dots b(1) e_\alpha$ is called untouched if no $\delta(l)$ with $s < l \leq k$ contains it.

We shall consider the lattices of partitions of words onto subwords. Let some the word $\beta = \rho\gamma\kappa$ is obtained from the word $\alpha = \rho\delta\kappa$ by the substitution $\delta \rightarrow \gamma$. Let also a partition G of α be given. We call a partition $G(\beta)$ of β the partition induced by G and the substitution

$\delta \rightarrow \gamma$ if following condition holds. If block I of G belongs to either ρ or κ then it is also a block of $G(\beta)$. The symbols of γ form one block together with all symbols of the blocks I (not belonging to δ) of G intersecting with δ .

Now let a partition G of β be given. We call a partition $G(\alpha)$ of α the partition induced by G and this substitution if following condition holds. If block I of G belongs to either ρ or κ then it is also a block of $G(\alpha)$. The symbols of δ form one block together with all symbols (not belonging to γ) of the blocks I of G intersecting with γ .

We define now inductively the set of partitions G_s of partitions of the words $\alpha_s = b(s) \dots b(1)\alpha$, $s = 0, 1, \dots, n$, where $\alpha = \alpha_0, \alpha_k = \alpha$. G_0 is the partition of α_0 onto n separate symbols. G_{s+1} is the partition of α_{s+1} induced by the substitution $\delta(s) \rightarrow \gamma(s)$. Denote $G_{s,0} = G_s$. If the partition $G_{s+1,p}$ of α_{s+1} is defined then $G_{s,p+1}$ is defined as the partition of α_s induced by the substitution $\delta(s) \rightarrow \gamma(s)$.

We need the partition $G_{0,k}$. Its blocks are at the same time the blocks of the partition of the interval $[1, n]$. We call them clusters with respect to (α, I_k, J_k) .

We call nonzero term of the expansion connected (for fixed α and (I_k, J_k)) if the partition $G_{0,k}$ consists of only one cluster.

Consider the contribution c_I of some cluster I . It depends only on its length m

$$c_I = c(m) = \sum_{\alpha: |\alpha|=m} \sum_{k=0}^{\infty} \sum_{(I_k, J_k)} \frac{(-\beta)^k}{k!} (e_{\alpha}, a_{i_k}(j_k) \dots a_{i_1}(j_1) e_{\alpha})$$

where the last sum is over all connected (α, I_k, J_k) . We have the cluster expansion for $z(N)$

$$z(N) \doteq \sum_{\alpha: |\alpha|=N} z(\alpha) = \sum c_{I_1} \dots c_{I_p}$$

where the sum is over all partitions on consecutive intervals. To prove this formula take the ordered array $\vec{m} = (m_1, \dots, m_p)$ of positive integers such that $m_1 + \dots + m_p = k$ and denote $\sum_{(I_k, J_k)}^{\vec{m}}$ the sum over all I_k, J_k such that the numbers of substitutions touching the consecutive subwords $\alpha_1, \dots, \alpha_p$, are correspondingly m_1, \dots, m_p . Then

$$\sum_{(I_k, J_k)}^{\vec{m}} z(\alpha, (I_k, J_k)) = \frac{k!}{m_1! \dots m_p!} \sum_{(I_{m_1}, J_{m_1})} z(\alpha_1, (I_{m_1}, J_{m_1})) \dots \sum_{(I_{m_p}, J_{m_p})} z(\alpha_p, (I_{m_p}, J_{m_p}))$$

We have also the cluster estimate

$$k(I) < c_1 (C\beta)^{|I|}$$

It follows that $\log z(N) \sim cN$. Thus

$$\log Z_N \sim \log \left[z(N) \left(1 + \frac{z(N-1)}{z(N)} + \dots \right) \right] \sim cN$$

Remark 2 Introduce the trivial substitutions $s \rightarrow s$ for each symbol $s \in S$ and denote $a(s; j)$ the corresponding quantum substitutions. Let $P_{=N}$ be the orthogonal projector onto the space $\mathcal{H}_{=N} = \mathcal{H}_N \ominus \mathcal{H}_{N-1}$. The cosmological term is defined as

$$\mu H^0 = \mu \sum_{N=0}^{\infty} N P_{=N} = \mu \sum_{s \in S} \sum_j a_s(j), \mu > 0$$

Note that for Hamiltonians with the cosmological term

$$H_N + \mu H^0$$

the limiting state exists for μ sufficiently large as the partition function is finite. It is natural to expect that there exists $\mu_{cr} = \mu_{cr}(\beta)$ such that for $\mu < \mu_{cr}$ the limiting state does not exist, but exists for $\mu > \mu_{cr}$. In most cases one can expect that either $\log Z_N \sim cN$ or it is constant. It could be interesting to know the cases when other possibilities occur. For example if $\beta = 0$ and S consists of one symbol only, then $\log Z_N \sim \log N$.

2.4 Classical space via renormalization

Assume β to be small as earlier and let us look at the "support" of $\langle \cdot \rangle_{\beta, N}$. More exactly, let \mathbf{C} be the commutative C^* -algebra, generated by multiplication (on bounded functions) operators in $\mathcal{H} = l_2(L)$. By restricting the state $\langle \cdot \rangle_{\beta, N}$ on the C^* -subalgebra $\mathbf{C}_N = \mathbf{C} \cap \mathbf{A}_N$, one gets the measure $\mu_{\beta, N}$ on the set of all words of length not exceeding N . One can show that as $N \rightarrow \infty$ the support of the measure $\mu_{\beta, N}$ lies on the words of length of order N .

Quantum Spin Systems We introduce some notation for quantum spin systems. Classical spin system on Z is a special probability measure on the set of configurations S^Z , that is functions on the "space" Z with values in S . The space Z has an additive group structure and acts on S^Z as a group of translations. The set of all words does not have such "space structure" but we shall show how the space (here it is Z), the quasilocal algebra on this space, and a KMS state on this quasilocal algebra, can emerge from a KMS-state on \mathbf{A} .

Consider classical spin configurations in a finite volume (that is the set $S^{[-n, n]}, [-n, n] \subset Z$) as words of length $2n+1$. The Hilbert space for the corresponding quantum spin system is

$$\mathcal{K}_{2n+1} = \otimes_{i=-n}^n \mathcal{K}(i)$$

where $\mathcal{K}(i)$ is the r -dimensional Hilbert space with basis $e_a, a = 1, \dots, r$. Consider the C^* -algebra of linear operators in \mathcal{K}_{2n+1} : $\mathbf{L}_{2n+1} = \mathbf{W}_{-n} \otimes \dots \otimes \mathbf{W}_n$, where \mathbf{W}_i are $r \times r$ -matrix algebras. The quasilocal quantum spin algebra \mathbf{L} is the norm closure of the local algebra $\mathbf{L}^0 = \cup \mathbf{L}_{2n+1}$.

Consider the Hilbert space $\mathcal{H}_{=n} = \mathcal{H}_n \ominus \mathcal{H}_{n-1} \subset \mathcal{H}$, generated by all $e_\alpha, |\alpha| = n$. Then \mathcal{K}_{2n+1} can be naturally identified with $\mathcal{H}_{=2n+1}$ by $\theta_{2n+1} : e_\alpha \in \mathcal{H}_{=2n+1} \rightarrow e_{a(1)} \otimes \dots \otimes e_{a(2n+1)} \in \mathcal{K}_{2n+1}$ if $\alpha = a(1) \dots a(2n+1)$.

Remark 3 If $|\delta_i| = |\gamma_i|$ for all i then the subspaces $\mathcal{H}_{=n}$ are invariant and thus we get quantum spin system Hamiltonians. In fact any quantum spin system Hamiltonians with finite range interaction can be obtained as particular cases of the Hamiltonians on quantum grammars by adjusting λ_i appropriately.

Consider the C^* -algebra \mathbf{M}_n of linear operators in $\mathcal{H}_{=n}$. Consider the isomorphism $\chi_{2n+1} : \mathbf{L}_{2n+1} \rightarrow \mathbf{M}_{2n+1}$ induced by θ_{2n+1} .

Consider the embeddings $\phi_n : \mathbf{M}_{2n+1} \rightarrow \mathbf{A}$, given for $M \in \mathbf{M}_{2n+1}$ by

$$\phi(M)e_\alpha = Me_\alpha, |\alpha| = 2n+1; \phi(M)e_\alpha = 0, |\alpha| \neq 2n+1$$

Consider some positive linear functional ω on \mathbf{A} . Then $\omega' = \omega \circ \phi_{2n+1} \circ \chi_{2n+1}$ is a positive linear functional on \mathbf{L}_{2n+1} . By normalizing we get the state $\langle \cdot \rangle_{2n+1} = Z_{2n+1}^{-1} \omega'(L)$ on \mathbf{L}_{2n+1} , $Z_{2n+1} = \omega'(1(\mathbf{L}_{2n+1}))$, where $1(\mathbf{L}_{2n+1})$ is the unit matrix in \mathbf{L}_{2n+1} . Consider the limiting state on the quasilocal algebra \mathbf{L}

$$\langle \cdot \rangle_{\beta, Z} = \lim_{n \rightarrow \infty} \langle \cdot \rangle_{2n+1}$$

if the limit exists.

Theorem 3 If β is small enough then the state $\langle \cdot \rangle_{\beta, Z}$ exists and is a KMS state on the quantum spin algebra \mathbf{L} .

Proof. Existence of the limiting state can be proven by cluster expansions. We have two representations

$$Z_{2n+1} = \omega'(1(\mathbf{L}_{2n+1})) = \sum c_{I_1} \dots c_{I_p}$$

and for some $A \in \mathbf{L}_{2n+1}$ with support in $[r, s] \subset [-n, n]$

$$\omega'(1(\mathbf{L}_{r+n}) \otimes A \otimes 1(\mathbf{L}_{n-s})) = \sum_{-n \leq m \leq r, p \leq n-s} \omega'(1(\mathbf{L}_{m+n})) c_{m,l}(A) \omega(1(\mathbf{L}_{n-l}))$$

The first representation was proved earlier, the second can be proved similarly. From these two representations the convergence to the limiting state follows by standard techniques, see [5]. It also follows from the cluster expansion that the limiting state is faithful, that is positive for positive elements. Thus in the GNS representation $(\mathfrak{M}, \pi, \Omega)$ the cyclic vector Ω is separating. Then Tomita-Takesaki theory defines a modular automorphism group of the von Neumann algebra and the limiting state is the KMS with respect to the modular group of automorphisms.

Remark 4 From the cluster expansion one could get more. This KMS state is limit of the states in finite volumes. Thus one could ask about the effective Hamiltonian for the resulting quantum spin system. The effective hamiltonian H_{eff} of this quantum spin system has non-finite multi-particle potential, that is

$$H_{eff} = \sum_{i \in Z} \sum_I \tau^i(\Phi_I)$$

where τ is the shift on 1 in the spin quasilocal algebra on Z and the second sum is over all intervals I containing 0. Moreover, for all I we have

$$|\Phi_I| \leq C\beta^{|I|}$$

for some $C > 0$.

We will not prove the statement of this remark.

Remark 5 *Note that the space structure can be obtained in different ways, using different embeddings. For example, one can get a quantum spin system on Z_+ (and the space then will be Z_+), using the isomorphism*

$$\mathcal{H}_n \rightarrow \otimes_{i=0}^{n-1} \mathcal{K}(i)$$

3 Quantum graph grammars

3.1 Definitions

Labelled spin graph $\alpha = (G, s)$ is a graph G with given set of vertices $V = V(G)$ and a function $s : V \rightarrow S$, where S is the spin space. Further on we assume it to be finite. Two labelled graphs are said to be equivalent (isomorphic) if they are isomorphic as graphs and the isomorphism respect spins. Equivalence classes are called (unlabelled) spin graphs. There are many other names for spin graphs: in physics spin graphs are referred as spin networks, in computer science they are called also marked graphs etc.

We remind definitions from [2].

Definition 1 *The substitution (production) $Sub = (\Gamma, \Gamma', V_0, \varphi)$ is defined by two "small" spin graphs Γ and Γ' , subset $V_0 \subset V = V(\Gamma)$ and mapping $\varphi : V_0 \rightarrow V' = V(\Gamma')$, either of Γ and Γ' can be empty.*

A transformation $T = T(Sub)$ of a spin graph α , corresponding to a given substitution Sub , is defined in the following way. Fix an isomorphism $\psi : \Gamma \rightarrow \Gamma_1$ onto a spin subgraph Γ_1 of α . Consider nonconnected union of α and Γ' , delete all links of Γ_1 , delete all vertices of $\psi(V) \setminus \psi(V_0)$ together with all links incident to them, identify each $\psi(v) \in \psi(V_0)$ with $v' = \varphi(v) \in \Gamma'$. The function s on $V(G) \setminus V(\Gamma_1)$ is inherited from α and on $V(\Gamma')$ - from Γ' . We denote the resulting graph by $\alpha(Sub, \psi)$.

The graph grammar is a finite set of substitutions $Sub_i, i = 1, \dots, m$. We call a graph grammar local if the Γ 's corresponding to all Sub_i are connected. The language $L(\alpha_0, \{Sub_i\})$ is the set of all spin graphs which can be obtained from some initial spin graph α_0 by applying transformations, corresponding to $Sub_i, i = 1, \dots, m$, arbitrary number of times in arbitrary order. More exactly, $\alpha_0 \in L(\alpha_0, \{Sub_i\})$ and if $\alpha \in L(\alpha_0, \{Sub_i\})$ then $T\alpha \in L(\alpha_0, \{Sub_i\})$ for arbitrary $T = T(Sub_i)$.

The definition of a quantum graph grammar is similar to that of the quantum grammar. Let \mathfrak{A} be a class of spin graphs, invariant with respect to the substitutions of the given grammar, for example $\mathfrak{A} = (\alpha_0, \{Sub_i\})$. Let $\mathcal{H} = \mathfrak{A}$ be the Hilbert space with the orthonormal basis e_α numerated by all spin graphs from \mathfrak{A} : $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. For each spin graph α and each substitution $Sub_i, i = 1, \dots, m$, we enumerate somehow all isomorphisms $\psi : \Gamma \rightarrow \Gamma_1$ as $\psi_1, \dots, \psi_{k(\Gamma)}$. Denote $a_i(j)$ the operator in \mathcal{H} by $a_i(j)e_\alpha = e_{\alpha(Sub_i, \psi_j)}$ if ψ_j exists, that is if $j \leq k(\Gamma)$, and 0 otherwise. Again we assume that together with the substitution $\delta_i \rightarrow \gamma_i$ also its "inverse" substitution $\gamma_i \rightarrow \delta_i$ belongs to Sub and the Hamiltonian is

$$H = \sum_{i=1}^r \sum_j \lambda_i a_i(j)$$

if $\lambda_i = \bar{\lambda}_j$ in case $\delta_j = \gamma_i, \gamma_j = \delta_i$. Note that the enumeration in j has only notational purpose, because the Hamiltonian is symmetric with respect to j .

3.2 Examples

We give here only two simplest examples.

3.2.1 Mean field evolution on graphs

There are no spins in this example. We consider 4 substitutions:

- Sub_1 is defined by Γ consisting of one vertex only, $V_0 = V(\Gamma)$, Γ' consisting of two vertices connected by a link. The mapping ϕ just fixes one of these vertices. Then the corresponding transformation consists of choosing a vertex v of the graph G , appending a new vertex v_{new} and connecting v and v_{new} by a link.
- Sub_2 is the inverse substitution, that is we take a link having at least one vertex of degree one and delete it.
- Sub_3 consists in appending a link between two chosen vertices.
- Sub_4 consists in just deleting a link.

This graph grammar is obviously nonlocal. The graphs can be non-connected and we still denote them α .

Seladjointness We shall see now that a reasonable choice of the constants is

$$B = \sum_j (\lambda_1 a_1(j) + \frac{1}{N} \lambda_2 a_3(j)), H = B + B^* = \lambda_1 \sum_j (a_1(j) + a_2(j)) + \frac{1}{N} \lambda_2 \sum_j (a_3(j) + a_4(j))$$

where N is the number of vertices in α and $\lambda_1, \lambda_2 \geq 0$. Note that finite linear combinations of e_α are C^∞ -vectors for L . Denote this set by $D(L)$.

Lemma 3 *H is essentially selfadjoint on $D(L)$.*

Proof. We shall prove that each vector from $D(L)$ is an analytic vector of L , that is

$$\sum_{k=0}^{\infty} \frac{\|H^k \phi\|}{k!} t^k < \infty$$

for some $t > 0$. Note that

$$\|B\|_{\mathcal{H}_N} = \lambda_1 N + \frac{\lambda_2 N^2}{N} = (\lambda_1 + \lambda_2)N$$

Here \mathcal{H}_N is generated by e_α , $V(\alpha) \leq N$. Note also that it is sufficient to take $\phi = e_\alpha$ and

$$H e_\alpha = \sum_{\beta} e_\beta$$

where $V(\beta) \leq V(\alpha) + 1$. The proof then is quite similar to the one for the quantum grammars.

3.2.2 Dual quantum evolution of two dimensional complexes

Here we consider a more physical example corresponding to the pure quantum gravity, where the quantum space has dimension 2 and the time is classical and has dimension 1. Consider the set \mathfrak{T} of equivalence classes of triangulations T of closed oriented compact surfaces $S = S_\rho$ of arbitrary genus $\rho = \rho(T)$, $N = N(T)$ is the number of triangles in T . A triangulation T is defined by a pair $(G; \phi : G \rightarrow S)$ where G is a graph and ϕ is its smooth embedding into S . Two triangulations T and T' are equivalent if there is a homeomorphism $\phi : S \rightarrow S'$ such that vertices of G go to the vertices of G' , edges to edges, triangles to triangles.

It is more convenient to consider dual graphs $\Gamma = G^*$, the vertices of Γ correspond to the triangles of G . Two vertices of the dual graph are connected by a link iff the corresponding triangles have common edge. Thus each vertex of Γ has degree 3. Then the number N of vertices of Γ is even. The set \mathfrak{T} of equivalence classes can be described equivalently in a purely combinatorial way in terms of dual graphs, see [37]. Consider the set \mathfrak{G} of graphs Γ with an additional structure. Each Γ has $N = N(\Gamma)$ vertices, each of degree 3, N is even. The additional structure on this graph is defined as follows: for each vertex the cyclic order of its edge-ends (legs) is fixed. It is not difficult to see that there is one-to-one correspondence between \mathfrak{G} and \mathfrak{T} . In one direction it is trivial: take an embedding $\Gamma \rightarrow S$ and fix orientation of S . Then choose say the clockwise order of the edge-ends in each vertex.

Let K be the set of edge-ends of Γ , it has then $3N$ elements. There are two permutations on K : the first one P consists of $V = N$ cycles of length 3 and the second one I consists of $E = \frac{3N}{2}$ cycles of length 2. Then vertices of Γ can be identified with cycles of the permutation P , edges (links) of Γ can be identified with cycles of the permutation I , faces of Γ can be identified with cycles of the permutation PI .

We introduce the Hilbert space $\mathcal{H} = l_2(\mathfrak{G})$ with the basis e_Γ . The evolution is defined as follows. For given $K = K(\Gamma)$ append 6 new elements, thus 2 cycles of length 3. Choose 3 edges j_1, j_2, j_3 (that is the cycles of length 2) in K , cut them thus getting 6 other edge-ends, and reconnect 12 edge-ends so that the resulting graph were connected. The reconnection is done via some rule π (depending on the set of 12 edge-ends). This will give the linear operator (quantum substitution) $a_\pi(j_1, j_2, j_3)$.

The resulting Hamiltonian is defined on the subspace \mathcal{H}_N generated by the graphs Γ with N vertices as follows

$$H = \frac{\lambda}{N^2}(B + B^*), B = \sum_{\pi} \sum_{j_1, j_2, j_3} a_\pi(j_1, j_2, j_3)$$

where j_1, j_2, j_3 is an arbitrary unordered array of 3 links. The adjoint term $B^* = \sum_{\pi} \sum_{v_1, v_2} b_\pi^*(v_1, v_2)$ describes the deletion of 2 vertices, v_1, v_2 is an unordered array of 2 vertices, π describes how the the remaining edge-ends are to be reconnected.

Note that the graphs here are not labelled. Thus one should be accurate with the automorphisms. Remind that almost all 3-regular graphs do not have nontrivial automorphisms.

It leaves invariant the symmetrical subspace \mathcal{H}_{symm} of the Hilbert space, that is the space of functions of L depending only on the number of triangles $N = N(L)$ and on the genus $\rho = \rho(L)$. Note that \mathcal{H}_{symm} is isomorphic to Z_+^2 . We shall study the spectral properties of this Hamiltonian in another paper.

4 Comments

4.1 Context Free Grammars

There are a lot of beautiful reviews on quantum computation now, see [35, 34, 33]. Quantum analogs of the standard computer science objects are quantum Turing machines, quantum circuits, quantum automata, quantum cellular automata etc. Our definition of a quantum grammar resembles partially each of them. That of a quantum cellular automaton, but where the lattice is a quantum object changing in time. Each term of the series expansion constitutes a transformation defined by a quantum circuit. For higher dimension we have a quantum analog of Kolmogorov-Uspenskij algorithms, or a quantum analog of the graph grammars.

The operators $a_i(j)$ in our definition are homogeneous (do not depend on j) but it is easy to consider inhomogeneous analog, taking the set of substitutions dependent of j . Then the time evolution of a quantum grammars can also be looked at as a general quantum circuit.

In the computer science there are some peculiarities in the definition of grammar. The alphabet S is the union $S = T \cup W$ of two nonintersecting alphabets: terminals T and non-terminals (variables) W . The substitutions (productions) $\alpha_i \rightarrow \beta_i, i = 1, \dots, m$, are such that each α_i contains at least one symbol from W . The quantum grammar is defined by the set of numbers $\lambda_i = \lambda(\alpha_i \rightarrow \beta_i)$, which are assumed to be real. In other words by the linear

operator

$$L = \lambda_i \sum a_i(j)$$

We prefer to use continuous time. If L is not assumed to be symmetric then one can define context free grammars, see [8] (but there are no nontrivial symmetric context free grammars). A context free grammar is one where all α_i have length 1, that is they are variables. Random context free grammars were studied in [1] in a more general situation when there is no subdivision of the alphabet.

For discrete time, which is assumed in [8], there are several ways to define the evolution, i.e. the derivation. Discrete time analog of our definition could be naturally given in a parallel form, that is all possible substitutions are done for the word at the moment. This is easy to do for context free grammars but not in more general cases. This is one of the reasons to use continuous time what we do here.

Let \mathcal{H}_{term} and \mathcal{H}_{var} be the Hilbert subspaces of \mathcal{H} defined by the corresponding parts of the alphabet S , and P_{term}, P_{var} are the orthogonal projections on these subspaces. The derivation is the mapping

$$\lim_{t \rightarrow \infty} P_{term} e^{itL} P_{var} : \mathcal{H}_{var} \rightarrow \mathcal{H}_{term}$$

Otherwise speaking we start with some word from W^* (the set of words over W), even with the symbol from W , and stop each time when all symbols in the resulting word are terminal. Note that e^{itL} is the identity on \mathcal{H}_{term} .

Existence of the dynamics can be proved quite similarly even in the non-symmetric case.

4.2 Spectrum

We saw already that the lattice models of statistical physics and quantum field theory constitute a particular case of the models on quantum lattices. The most interesting question is the study of the spectrum of such models: whether it has particles, scattering etc. We show below that the spectrum have some new features even for the simplest models.

We already mentioned that the derivation for the grammar is described by the operator $P \exp itH$, only large t are interesting for us. If one knows that H is unitary equivalent to H_0 for some simple H_0 , that is $H = UH_0U^{-1}$, then the operator $P \exp itH$ reduces to $P_U \exp itH_0, P_U = UP U^{-1}$. If H describes something like interacting infinite particle system then H the H^0 can be the corresponding free hamiltonian describing free quasiparticles. That is why spectral properties of H are related to the derivation in grammars.

Note that e_\emptyset is a zero eigenvector of H . One could expect that the rest of the spectrum of such operators should be similar to the spectra of many particle systems. In particular one could expect that H is unitary equivalent to a free hamiltonian in a Fock space over some one-particle subspaces. One could expect also that among these particles some correspond to quanta of space and some - to quanta of matter fields. Could one find an exact formulation of this statement ?

Detailed study of the spectrum of H is necessary for this, and we shall do in another paper, here we only give simplest examples. In the rest of the paper we shall follow another

idea: under some scaling we get a classical space (here the lattice Z) and quantum spin system on it. This scaling destroys thus the quantum character of space.

1. (Quantum spin systems) We say that the hamiltonian H is space (or lattice) conserving if $|\delta_i| = |\gamma_i|$ for all i . Space conserving operators can be reduced to quantum spin systems, as we shall see below. In this case there are only particles corresponding to matter.
2. (One-particle space) Let $r = 1$, that is the alphabet consists of one symbol a . Consider two substitutions $1 : a \rightarrow aa, 2 : aa \rightarrow a, \lambda_1 = \lambda_2 = \lambda$, and the Hamiltonian

$$H = \lambda \sum_{j=1}^{\infty} (a_1(j) + a_2(j))$$

with real λ . Then the Hilbert space \mathcal{H} is isomorphic to $l_2(Z_+)$, because the word $aa...a$ can be identified with its length minus 1. The Hamiltonian is unitary equivalent to Jacobi matrix

$$(Hf)(n) = \lambda(n-1)f(n-1) + \lambda n f(n+1)$$

We shall call this operator a one-particle operator, this "particle" is natural to associate with a space quanta. Here the space evolution is the simplest one (due to one dimension): expansion and compression (in each point). We shall find its spectrum in another paper.

3. Consider $S = \{a, w\}$ and the following substitutions

$$a \rightarrow aa, aa \rightarrow a, aw \rightarrow wa, wa \rightarrow aw$$

The subspace \mathcal{H}_1 generated by words with exactly one symbol w is invariant. This hamiltonian can be interpreted as a mixture of the previous pure space hamiltonian H_1 and the discrete laplacian H_2 , the free nonrelativistic one-dimensional Schroedinger operator in l_2 on a finite set. Similarly, the subspace \mathcal{H}_2 with exactly two symbols w corresponds to two matter particles.

4. (Noncommutative Fock space) Let $S = \{a, b\}$ and consider 4 substitutions

$$1 : a \rightarrow aa, 2 : aa \rightarrow a, 3 : b \rightarrow bb, 4 : bb \rightarrow b$$

Here there are two invariant one-particle spaces $\mathcal{H}_a, \mathcal{H}_b$. For example, \mathcal{H}_a is generated by words $a, a^2 = aa, \dots, a^n, \dots$. There are two invariant two-particle spaces $\mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b, \mathcal{H}_{ba} = \mathcal{H}_b \otimes \mathcal{H}_a$. For example, \mathcal{H}_{ab} is generated by words $a^k b^l, k, l > 0$. In general for each even n there two invariant $2n$ -particle spaces $\mathcal{H}_{(ab)_n}$, generated by words $a^{k_1} b^{l_1} \dots a^{k_n} b^{l_n}$, and $\mathcal{H}_{(ba)_n}$. Similarly for odd n there are two invariant $(2n+1)$ -particle spaces $\mathcal{H}_{b(ab)^n}, \mathcal{H}_{(ab)^n a}$. For arbitrary $r > 2$ with substitutions $i \rightarrow ii, ii \rightarrow i$ for each $i = 1, 2, \dots, r$ we have $r(r-1)^{n-1}$ n -particle spaces. Thus, the standard spectrum of

tensor products has the corresponding multiplicities. This example supports the name "one-particle" in the first example, because we get here a Fock space OVER these two one-particle spaces.

This shows a rich structure of the introduced Hamiltonians.

4.3 Short overview of evolution types

We give here a very short overview of other papers where related dynamics of discrete structures were considered. Note that most papers have more geometric and algebraic aspect than analytic one. In our paper we considered mainly analytic problems.

Deterministic evolution Deterministic evolution of words is one of the main subjects of the computer science. In computer science marked graphs and their deterministic evolution were known since Kolmogorov-Uspenskij paper [4]. Now there is a large field in computer science, which studies graph grammars - local dynamics of the marked graphs. In [20] deterministic evolution of classical spin systems on graphs is defined. The basic graph is fixed or taken randomly via random graph theory procedure, that is for fixed set of vertices each bond is drawn independently with some probability $0 < p < 1$.

Markov processes Random grammars were considered earlier in computer science context, as Markov processes. But questions related to the thermodynamic limit appeared only in [1, 2].

Unitary evolution and causal structure Such evolution is the main object in quantum computing in the computer science context and in quantum gravity in a physical context. The latter considers spin graph as a quantum object, thus one deals with the wave function on the set of all possible spin graphs. The square of the wave function defines a probability distribution on spin graphs.

Spin networks (graphs with spins, half-integers, living on the links, and some operators in the vertices) were introduced in physics by R. Penrose [17]. In physics now there are many variants of the quantum evolution, discussed in [26, 27, 30, 10, 15, 23, 24, 31] as well as in earlier papers, see [7]. Wider generalization are discrete complexes with a causal structure which could model Lorentzian structure on manifolds.

Completely positive semigroups This case is interesting due to inevitable noise coming from the environment of the quantum system. See discussion of these problems in [33].

Nonlinear Markov processes There can be transformations of probability measures on spin complexes, which cannot be reduced to a Markov process (that is they are not given by random point transformations) and they are not of quantum mechanical nature. Examples of such dynamics one can find in two-dimensional quantum gravity, see [3].

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